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On the impact of noise on quantum chaos

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bstract : Using the Langevin formalism, the impact of damping, noise and potential gradient on a particle in a double well potential has been udied in the quantum regime and results compared with the $\hbar \to 0$ classical limit. It has been shown that the effects of damping and noise on the stem are complementary in so far as the reduction of chaocity of the system is concerned, both opposing the potential derived force. Further, as wards the inducement of quantum chaos, quantum tunneling effects are shown to dominate. The limitations of the study, in particular, the efficacy f the Ehrenfest's theorem in the classical regime are also explored

eywords : Quantum chaos, Langevin equation, quantum noise

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Introduction

he impact of noise on the onset of chaos in classical systems s been extensively studied [1, 2]. However, such studies in uantum systems have been sparse, possibly, because the manating stochastic differential equations are not amenable to umerical solution. In fact, most of the study of quantum chaos as been confined to conservative systems. With the echnological advances in the fields of quantum optics, ryogenics, statistical thermodynamics *etc.* the study of open quantum systems has assumed great significance.

The presence of Langevin noise in classical systems has hown to suppress homoclinic crossing and to raise the omoclinic threshold [3], besides destroying the periodic points mbedded in the positive region of the Lyapunov exponent and odifying the period doubling route to chaos (by introducing aps in the period doubling sequence)[1, 4-6]. Studies on the erturbed Josephson junction have indicated shifting of the ifurcation points proportional to the square of the perturbing mplitude in the presence of noise, particularly near the period oubling threshold [7, 8]. Suppression of chaotic behavior in a ulling oscillator by a noisy perturbation has also been reported Analysis of dissipative nonlinear systems under the action periodic and random forces have revealed the existence of multiple maxima in the chaotic regime of the probability density function [4].

A majority of the above analysis are based on traditional methods of studying phase space dynamics (such as the K entropy [10], Lyapunov numbers [11], Melnikov function [12], the tests of Zaslavsky and Chirikov [13], Greene [14] and Mo [15], Toda [16], Brumer and Duff [17] and Pattanayak and Schieve [18]) which cannot directly be adopted into the realms of quantum mechanics in the absence of a standard phase space.

Various approaches to this problem involve, either the truncation of the infinite dimensional Hilbert space associated with the system using squeezed coherent states [19], adoption of a variationally restricted form of the Schrödinger equation [20], carrying out of calculations in the full Hilbert space and then projecting the same onto the classical phase space using an appropriate distribution [21] or the computation of an equivalent effective classical potential and thereafter working in the classical phase space [22].

Open quantum systems subject to damping/dissipation/ noise have not been extensively studied. For classical systems, the theory of stochastic processes provides a well founded basis for a phenomenological approach to the study of the

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master equation taking the form of the Fokker Planck equation on the assumption of the processes being Gaussian Markovian since there exists an immediate correlation between fluctuations and dissipation - no such counterpart exists in the quantum regime. Path integral techniques like the influence functional [23, 24], decoherence functional [25, 26] and the Quantum State Diffusion [27] have been used to study quantum dissipative systems. Influence functionals [24] of Brownian motion with dissipation have shown that the motion reduces to the Fokker Planck motion in the limit $\hbar \to 0$. Decoherence studies [25, 26], which result in the appearance of inherent noise in the system due to coarse graining, have shown a sharp transition in the entropy production rate between reversible and dissipative regimes Intimate connection between the decoherence histories and the quantum trajectory derived by the quantum state diffusion method using a moving basis has been established for the quantum version of the Duffing oscillator [28]. The homomorphic dynamical map, random matrices and solitons have also been used for the study of quantum systems.

The quantum version of the Langevin equation [29, 30] has recently appeared in numerous literature to study the semiclassical dynamics of both conservative and dissipative quantum systems using the system - bath model In the present paper, we study the phase space dynamics of an open quantum system using the Langevin formalism in the presence of a noise term and examine its impact on the chaotic evolution, comparing it with the $\hbar \to 0$ classical limit, where noise is indicated to suppress the onset of chaos. Using the technique of squeezed coherent states, we shall truncate the infinite dimensional Hilbert space of the problem, and through the Taylor series expansions, derive the phase space dynamics for the spread of the waves, in addition to those of the centroid We also study in detail the efficacy of Ehrenfest's theorem in describing system dynamics in the $\hbar \to 0$ classical limit.

2. Formulation of the problem

We consider here the simplest model of a dissipative quantum mechanical system consisting of a Brownian particle of unit mass subject to a time independent one dimensional bounded potential $V(Q)$ and interacting with the environment.

If we simulate the environment as a bath of n harmonic oscillators obeying exactly Gaussian statistics and in the case of a separable, strictly linear interaction between the particle and the environment, we may write the Hamiltonian of the composite system as

$$H_{tot} = H_{part} + \frac{1}{2}\sum \left| \frac{p_\alpha^2}{m} + m_\alpha \omega_\alpha^2 \right| q_\alpha \quad \frac{c_\alpha Q}{m_\alpha \omega_\alpha^2} \quad \Big| , \qquad (1)$$

where H_{part} is the Hamiltonian of the isolated Brownian particle, m_α and ω_α are respectively the mass and frequency of the α^{th} oscillator in the bath and c_α is the coupling constant between the α^{th} oscillator and the Brownian particle

The equal time canonical commutation relations appended to (1), in natural units $c = \hbar = 1$ are

$$[P, p_\alpha] = [P, q_\alpha] = [Q, p_\alpha] = [Q, q_\alpha] = 0,$$

$$[p_\alpha, p_\beta] = [q_\alpha, q_\beta] = 0$$

and

$$[P, Q] = [p_\alpha, q_\alpha] = -i.$$

The Heisenberg equation of motion for an arbitrary operator is obtained by commuting it with the total Hamiltonian Hence these equations of motion of the environment variables are given by

$$\dot{q}_\alpha = \frac{i}{\hbar}[H_{tot}, q_\alpha] = \frac{p_\alpha}{m_\alpha}$$

and

$$\dot{p}_\alpha = \frac{i}{\hbar}[H_{tot}, p_\alpha] = c_\alpha - m_\alpha \omega_\alpha^2 q_\alpha$$

which admit the following solutions :

$$q_\alpha(t) = \frac{1}{2} \begin{matrix} q_\alpha(t_0)\left[e^{-i\omega_\alpha(t-t_0)} + e^{i\omega_\alpha(t-t_0)}\right] + \\ \cdot\, p_\alpha(t_0)\left[e^{-i\omega_\alpha(t-t_0)} + e^{i\omega_\alpha(t-t_0)}\right. \end{matrix}$$

$$+ \frac{ic_\alpha}{m\ \omega} \int^t \left[e^{-i\omega_\alpha(t-t')} - e^{i\omega_\alpha(t-t')}\right] Q(t')\,dt'$$

and

$$p_\alpha(t) = \frac{1}{2} \begin{cases} p_\alpha(t_0)\left[e^{-i\omega_\alpha(t-t_0)} + e^{i\omega_\alpha(t-t_0)}\right] + \\ i m_\alpha \omega_\alpha q_\alpha(t_0)\left[e^{-i\omega_\alpha(t-t_0)} + e^{i\omega_\alpha(t-}\right. \end{cases}$$

$$+ c_\alpha \int^t \left[e^{-i\omega_\alpha(t-t')} + e^{i\omega_\alpha(t-t')}\right] Q(t')\,dt'.$$

By commuting the particle operators with the Hamiltonian (1), we get their respective equations as

$$\dot{Y} = \frac{i}{\hbar}\left[H_{part}, Y\right] + \frac{i}{2\eta}\sum_{\alpha=1}^{N} \begin{matrix} \frac{c_\alpha^2}{m_\alpha \omega_\alpha^2}(Q[Q, Y] + [Q \\ -c_\alpha q_\alpha [Q, Y] \end{matrix}$$

$$= \frac{i}{\hbar}\left[H_{part}, Y\right] + \frac{i}{2\eta}\sum_{i}^{..} \quad Y, q_\alpha - \frac{c_\alpha Q}{m_\alpha \omega_\alpha^2}. \quad , c_\alpha Q$$

$$= \frac{i}{\hbar}\left[H_{part}, \dot{Y}\right] + \frac{i}{2\hbar}\sum_{\alpha=1}^{N} \left[Y, c_\alpha Q\right], q_\alpha \cdot \cdot \frac{c_\alpha Q}{m_\alpha \omega_\alpha^2} \tag{9}$$

On substituting the value of q_α from eq. (7) in eq. (9), we get

$$\frac{i}{\hbar}\left[H_{part}, \dot{Y}\right] - \frac{i}{2\hbar} \begin{matrix} [Q, Y], \xi(t) - \int_{t_0}^{t} f(t-t')\,\dot{Q}(t')\,dt' \\ \left\lfloor -f(t-t_0)\,Q(t_0) \right. \end{matrix} \tag{10}$$

where

$$\xi(t) = \frac{1}{2}\left\{\sum_{\alpha=1}^{N} m_\alpha \omega_\alpha^2 q_\alpha(t_0)\left[\begin{matrix} e^{-i\omega_\alpha(t-t_0)} + \\ e^{i\omega_\alpha(t-t_0)} \end{matrix}\right.\right.$$

$$+ i\omega_\alpha p_\alpha(t_0)\Big| \begin{matrix} -i\omega_\alpha(t-t_0) \;\_\rceil \\ e^{i\omega_\alpha(t-t_0)} \end{matrix} \tag{11}$$

$$f(t) = \sum_{\alpha=1}^{..} m_\alpha \omega_\alpha^2 \cos(\omega_\alpha t). \tag{12}$$

Identifying Y as the canonical coordinate Q and the corresponding momentum P respectively of the Brownian particle and H_{part} as $\frac{P^2}{2} + V(Q)$, we get the following quantum Langevin equations for the system

$$Q(t) = P(t) \tag{13}$$

and

$$P(t) = -\left(V'(Q)\right) - \int_{t_0} f(t-t')\dot{Q}(t')dt' + \xi(t) - f(t-t_0)Q(t_0). \tag{14}$$

Now, if $f(t)$ is a rapidly decaying function, so that it goes to zero in a time scale which is much less than the time over which $Q(t')$ changes, then we can replace $\dot{Q}(t')$ by $\dot{Q}(t)$ in eq. (14). Further, if t is not close to t_0, we can take

$$f(t-t_0) \cong 0, \tag{15}$$

so that eqs. (13) and (14) reduce to

$$Q(t) = P(t) \tag{16}$$

and

$$\dot{P}(t) = -\left(V'(Q)\right) - \dot{Q}(t)\int^{t} f(t-t')dt' + \xi(t). \tag{17}$$

If we further approximate the frequency spectrum of the environment oscillators by a continuous frequency distribution $G(\omega)$, we can write

$$f(t) = \sum_{\alpha=1} m_\alpha \omega_\alpha^2 \cos(\omega_\alpha t) \rightarrow \int \cos(\omega t)G(\omega)\frac{dn(\omega)}{d\omega}d\omega, \tag{18}$$

where $G(\omega)d\omega$ is the number of modes in the frequency range ω and $\omega + d\omega$. In this continuum limit, $G(\omega)\frac{dn(\omega)}{d\omega}$ represents the effect of the dissipation of energy through the modes in the frequency range ω and $\omega + d\omega$. We can make a further simplification of eq. (18) in the first Markov approximation, wherein $G(\omega)\frac{dn(\omega)}{d\omega}$ is assumed constant. Then, it follows that

$$f(t) = \frac{2\gamma}{\pi}\int^{\infty} \cos(\omega t)d\omega = 2\gamma\delta(t), \tag{19}$$

where

$$\int^{\infty} G(\omega)\frac{dn(\omega)}{d\omega}d\omega = \frac{2\gamma}{\pi}.$$

Hence, in the Gaussian Markovian approximations, our equations of motion (13), (14) and (9) take the form:

$$\dot{Q}(t) = P(t), \tag{20}$$

$$\dot{P}(t) = -\left(V'(Q)\right) + \gamma P + \xi(t) \tag{21}$$

and

$$\dot{Y} = \frac{i}{\hbar}\left[H_{part}, Y\right] - \frac{i}{2\hbar}\left[[Q, Y], \xi(t) - \gamma\dot{Q}\right]_+. \tag{22}$$

$\xi(t)$ is a random noise function which obeys the following commutation relations :

$$\left[\xi(t), \xi(t')\right] = i\hbar\frac{d}{dt}f(t-t'), \tag{23}$$

$$\left[Y(t), \xi(s)\right] = \int^{t}\left[Y(t), Q(t')\right]\frac{d}{ds}f(s-t')dt', \tag{24}$$

$$\left[Y(t), x_\alpha(t_0)\right] = \frac{\omega_\alpha}{2i}\int^{t}\left[e^{-i\omega_\alpha(t'-t_0)} - e^{i\omega_\alpha(t'-t_0)}\right]\left[Y(t), Q(t')\right]dt' \tag{25}$$

and

$$\left[Y(t),\, p_\alpha(t_0)\right] = \frac{m_\alpha \omega_\alpha^2}{2i} \int_{t_0}^{t} \left[e^{-i\omega_\alpha (t'-t_0)} + e^{i\omega_\alpha (t'-t_0)} \right]$$

$$\times \left[Y(t),\, Q(t')\right] dt'. \tag{26}$$

In the first Markov approximation, the commutation relations (23) and (24) reduce to

$$\left[\xi(t), \xi(t')\right] = 2i\hbar\gamma \frac{d}{dt} \delta(t-t') \tag{27}$$

and

$$\left[Y(t), \xi(s)\right] = 2\gamma \frac{d}{ds} \left\{ u(t-s) \left[Y(t), Q(s)\right] \right\}, \tag{28}$$

where

$u(x) = 1,\ if\ x > 0,$

$\frac{1}{2}\ if\ \ x = 0,$

$= 0\ \ if\ \ x < 0.$

The quantum Langevin equations (20-22) are operator equations and are therefore, not directly amenable to numerical simulation. We shall now, attempt to eliminate the operator dependence of each of the constituents of the QLE.

3. Elimination of operator dependence of $\xi(t)$ [30]

For this purpose, we assume that the density operator (in the Heisenberg picture) factorizes into a direct product

$$\rho_{part} \otimes \rho_{em}\ .$$

Let $Y(t)$ be an operator in the Heisenberg picture and Y be the corresponding operator in the Schrodinger picture. Then, we can consistently define a quantity $\mu(t)$ by the equation

$$Tr_{part}\left[Y(t)\rho_{part}\right] = Tr\left[Y\mu(t)\right]. \tag{29}$$

Let e_i constitute a set of basis vectors in the phase space of the particle in the Schrödinger representation. Then, any particle operator Y can be expressed as a linear combination of these e_i. Further, let them be orthogonal with respect to the trace function *i.e.*

$$Tr_{part}\left(e_i^{\dagger} e_j\right) = \delta_{ij}\ . \tag{30}$$

Then, the operator Y of the particle can be written in this basis as

$$Y = \sum Tr_{part}\left\{e_i Y\right\} e_i^{\dagger}\ , \tag{31}$$

that

$$Ye_i = \sum Tr_{part}\left\{e_i^{\dagger} Y e_i\right\} e \tag{32}$$

Let $e_i(t)$ be the corresponding basis in the Heisenberg representation. In quantum mechanics, time evolution is unitary and all algebraic relations are preserved. Hence, if $Y(t)$ is the operator (in the Heisenberg representation) corresponding to Y (in the Schrodinger representation), we can write

$$Y(t)e_i(t) = \sum Tr_{part}\left\{e_j(t)^{\dagger} Y(t) e_i(t)\right\} e_j(t)\ ,$$

where $Tr_{part}\left[e_j(t)^{\dagger} Y(t) e_i(t)\right]$ has exactly the same values as in the Schrodinger representation *i e.* $Tr_{part}\left(e_j^{\dagger} Y e_i\right)$

Using the expression for Y given by eq. (31) and the corresponding expression for $Y(t)$, we get on substitution in (29)

$$Tr_{part}\left[e_i(t)\rho_{part}\right] = Tr_{part}\left[e_i \mu(t)\right], \tag{34}$$

which on using (31), gives the following explicit expression for $\mu(t)$:

$$\mu(t) = \sum_i Tr_{part}\left\{e_i(t)\rho_{part}\right\} e_i^{\dagger}\ . \tag{35}$$

Differentiation with respect to t yields

$$\dot{\mu}(t) = \sum_i Tr_{part}\left\{\dot{e}_i(t)\rho_{part}\right\} e_i^{\dagger}\ . \tag{36}$$

The value of $e_i(t)$ can be obtained by substituting $Y = \dot{e}_i(t)$ in eq. (22) and we have

$$e_i(t) = \frac{i}{\hbar}\left[H_{part}, e_i(t)\right] - \frac{i}{2\hbar}\left[\left[Q(t), e_i(t)\right], \xi(t) - \gamma Q(t)\right]_{+} \tag{37}$$

Now, consider the expression

$$Tr_{part}\left[Q(t)\ e_i(t)\xi(t)\rho_{part}\right],$$

$$= Tr_{part}\left\{\sum Tr_{part}\left[Q(t)e_i e_j^{\dagger}\right] e_j(t)\rho_{part}\right\}\xi(t),$$

$$= Tr_{part}\left\{\sum_j Tr_{part}\left[e_j^{\dagger} Q(t) e_i\right] e_j(t)\rho_{part}\right\}\xi(t). \tag{38}$$

The corresponding expression in $\dot{\mu}(t)$ is

$$\sum Tr_{part}\left\{\sum Tr_{part}\left[e_j^{\dagger} Q(t) e_i\right] e_j(t)\rho_{part}\right\} e_i^{\dagger}\xi(t)$$

$$= \sum Tr_{part}\left[e_j(t)\rho_{part}\right]e_i^{\dagger}Q(t)\xi(t)$$

$$= \mu(t)Q(t)\xi(t). \tag{39}$$

Similarly, the term corresponding to $Tr_{part}\left[e_i(t)Q(t)\xi(t)\rho_{part}\right]$ in $\dot{\mu}(t)$ may be written in the form $Q(t)\mu(t)\xi(t)$, the term corresponding to $Tr_{part}\left[Q(t)e_i(t)\dot{Q}(t)\rho_{part}\right]$ as $Q(t)\mu(t)\dot{Q}(t)$ and so on. On making these substitutions in eq. (37) and using the commutation relations between $Q(t)$, $\xi(t)$ and $\dot{Q}(t)$, we get

$$\dot{\mu}(t) = -\frac{i}{\hbar}\left[H_{part}, \mu(t)\right] + \frac{i}{2\hbar}\left[\left[\gamma\dot{Q}(t) - \xi(t), \mu(t)\right]_+, Q(t)\right], \tag{40}$$

$$\dot{Q}(t) = \frac{i}{\hbar}\left[H_{part}, Q(t)\right]. \tag{41}$$

In eq. (41), $\xi(t)$ appears as an anti-commutator. We define an operator $\eta(t)$ by the equation

$$\eta(t)\mu(t') = \frac{1}{2}\left[\xi(t), \mu(t')\right]_+ \tag{42}$$

for all t, t'. Then, we have

$$\left[\eta(t), \eta(t')\right]\mu = \frac{1}{4}\left[\left[\xi(t), \xi(t')\right], \mu\right]. \tag{43}$$

However, the commutator $\left[\xi(t), \xi(t')\right]$ is a c-number. Hence, $[\eta(t), \eta(t')] = 0$. By definition, multiplication of $\eta(t)$ with $\mu(t')$ is associative, so that $\eta(t)$ is equivalent to a c- number random function. In terms of $\eta(t)$, our Langevin equation (40) takes the form.

$$\dot{\mu}(t) = -\frac{i}{\hbar}\left[H_{part}, \mu(t)\right] + \frac{i\gamma}{2\hbar}\left[\dot{Q}(t), \mu(t)\right]$$

$$-i\hbar\eta(t)\left[\mu(t), Q(t)\right] \tag{44}$$

which, in the case when we identify $\mu(t)$ as $Q(t)$ and $P(t)$ respectively gives, using the commutation relations between Q and P,

$$\dot{Q} = P \tag{45}$$

$$\dot{P} = -(V'(Q)) + \gamma P + \eta(t), \tag{46}$$

where $\eta(t)$ is now a c-number quantity. However, Q and P are operators.

4. Elimination of operator dependence of Q and P [31]

To make further progress in our program of making the QLE amenable to simulation, we make a critical assumption that the Brownian particle is representable by a squeezed coherent state wave packet of the form :

$$\left|\Phi(\alpha, \beta)\right\rangle_{CS} = e^{S(\alpha)T(\beta)}\left|0\right\rangle, \tag{47}$$

where $e^{T(\beta)}$ is the unitary squeeze operator and $e^{T(\beta)}\left|0\right\rangle$ is the squeezed vacuum state. $e^{S(\alpha)}$ and $e^{T(\beta)}$ are unitary transformations generated by the linear and quadratic forms of the creation and annihilation operators a and $a^{\dagger}$ and are defined by

$$S(\alpha) = \left(\alpha a^{\dagger} - \alpha^* a\right) \tag{48}$$

and

$$T(\beta) = \frac{1}{2}\left(B a^{\dagger 2} - B^* a^2\right). \tag{49}$$

(In the context, a squeezed coherent state as above has been shown to be equivalent to a generalized Gaussian wavepacket derived on the basis of the time dependent variational approach [20] *i.e.*

$$\left|\psi_+, t\right\rangle_g = \frac{1}{(2\pi\hbar G)^{1/4}} e^{-\frac{1}{2\hbar}(Q-q)^2\left(\frac{1}{2G} - 2i\Pi\right) + \frac{i}{\hbar}p(Q-q) + \frac{i}{2\hbar}pq} \tag{50}$$

and

$$\left|\psi_-, t\right\rangle_g = \frac{1}{(2\pi\hbar G)^{1/4}} e^{-\frac{1}{2\eta}(Q-q)^2\left(\frac{1}{2G} + 2i\Pi\right) - \frac{i}{\eta}p(Q-q) - \frac{i}{2\eta}pq} \tag{51}$$

in Ref. [19]).

For the squeezed coherent states, the following expressions for the centroid and the spread of the wave packet *i.e.* the moments of Q and P are well known [19] :

$$\left\langle\tilde{Q}^{2m}\right\rangle = \frac{(2m)!(\hbar\mu)^n}{m!2^m} \tag{52}$$

$$\left\langle Q^{2m+1}\right\rangle = 0, \tag{53}$$

$$\left\langle\tilde{P}^2\right\rangle = \frac{\hbar\left(1+\alpha^2\right)}{4\mu} \tag{54}$$

and

$$\left\langle\tilde{P}\tilde{Q} + \tilde{Q}\tilde{P}\right\rangle = \hbar\alpha, \tag{55}$$

(where ~ denotes deviation of the respective variables from their expectation values). We can now generate a countably infinite number of moment equations around the centroid of the wave packet corresponding to the infinite dimensional Hilbert space of the problem by the Taylor's series expansion formula

$$F(Q)=\sum \frac{\partial^n F(Q)}{\partial Q^n}\bigg|_{Q=\langle Q\rangle}\frac{\tilde{Q}^n}{n!}$$ for the operators Q and P

Assuming an explicit representation for the potential function $V(Q)$ as :

$$V(Q)=-\frac{1}{2}aQ^2+\frac{1}{4}bQ^4+\frac{1}{6}cQ^6 \tag{56}$$

and using eqs. (52-55), we finally arrive at the following set of stochastic differential equations for the centroid and spread of the squeezed wave packet :

$$\dot{Q}=P, \tag{57}$$

$$\dot{P}=aQ-bQ^3-cQ^5-3bQ\hbar\mu-10cQ^3\hbar\mu$$
$$-15cQ\hbar^2\mu^2-\gamma P+\eta(t), \tag{58}$$

$$\dot{\mu}=\alpha\ , \tag{59}$$

and

$$\dot{\alpha}=\frac{\left(1+\alpha^2\right)}{2\mu}+2\mu\left(a-3bQ^2-10cQ^4\right)$$
$$-6\mu^2\left(b\hbar+5cQ^2\hbar\right)-30c\mu^3\hbar-\gamma\alpha. \tag{60}$$

In eqs. (57-60), Q and P are now the expectations of the operators Q, P *i.e.* are the coordinates of the centroid of the wavepacket in phase space and μ and α are indicative of the spread of the squeezed coherent state.

5. Numerical analysis

We trace out the time series and the phase space plots for the variables Q, P and the fluctuation variables μ, α representing the motion of the centroid of the wave packet and its spread respectively for various combinations of the values of c, γ taking $\hbar = 0.1$, $a = 10$, $b = 4$ with and without noise.

We find that in the classical limit $(\hbar \to 0)$, the Q, P equations decouple from the μ, α equations and we get two sets of first order differential equations. The motion therefore, is regular and periodic throughout, with trajectories in both potential wells (Figure 1(a1-b1)). On introduction of damping, for a certain initial period, the particle is able to move across the potential barrier at $Q = 0$ and execute motion in both the potential wells. Thereafter, the motion gets restricted to one well and the particle orbit also goes on shrinking. The long-term evolution of the system seems to move towards a point attractor. An increase in damping is seen to reduce this bifurcation point and increase the rate of shrinkage of the orbit (in our simulation exercise, the bifurcation point was seen to reduce from $t = 53.6$ when $\gamma = 0.002$ to $t = 18.4$ when $\gamma = 0.005$) (Figures 1(a2-b2, a3-b3)). The existence of this bifurcation may be attributed to the dissipation of energy from the system due to damping. With time evolution, the energy of the particle reduces below a certain threshold level and it is unable to cross the potential barrier. Accordingly, its motion becomes restricted to one well. The orbit shrinkage may also be due to dissipation of energy from the system due to damping. An increase in the steepness of the potential (monitored through the value of c) is seen to reduce the period of oscillation of the particle and also the width of the Q, P phase diagram. The bifurcation point is however, seen to increase (from $t = 53.6$ when $c = 0$, $\gamma = 0.002$ to $t = 61.0$ when $c = 1$, $\gamma = 0.002$) (Figures 1(a1-b1), 2(a1-b3)).

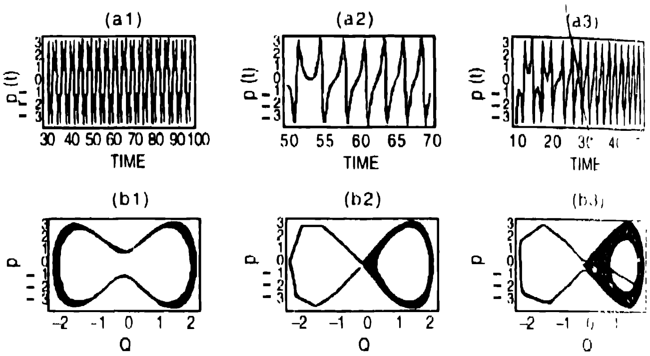


Figure 1. Time series and phase plots for Q, P without external noise and $\eta = 0$, $a = 10$, $b = 4$, $c = 0$ [$\gamma = 0$ (a1-b1)], [$\gamma = 0.002$ (a2-b2)], [$\gamma = 0.005$ (a3-b3)]

In the classical case $(\hbar \to 0)$, superposition of external noise does not effect the regularity of the motion. However, the particle motion gets confined to one well entirely, the point of zero velocity tends to move away from the origin, there is further increase in the oscillation frequency and the orbit shrinks faster (Figures 3 and 4). This indicates the complementary nature of external noise and damping, both opposing the potential derived

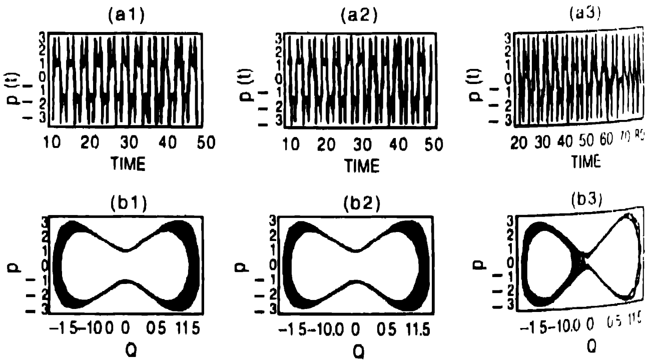
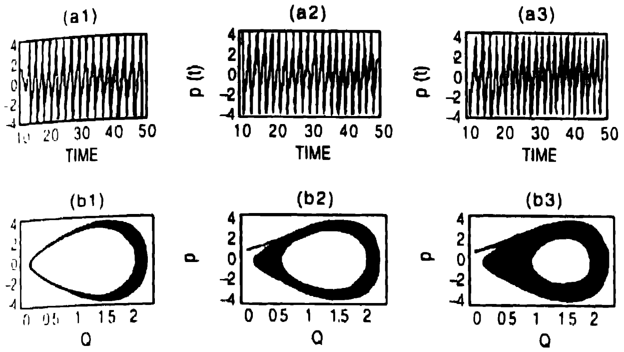
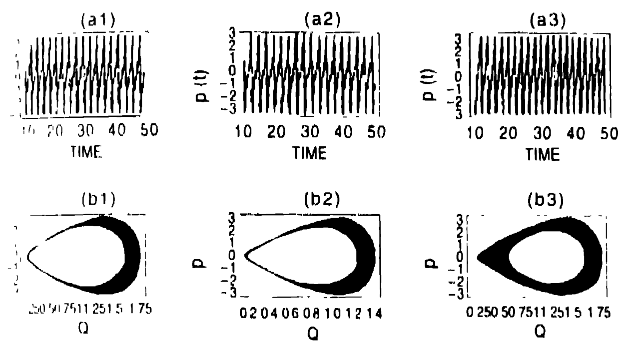


Figure 2. Time series and phase plots for Q, P without external noise and $\eta = 0$, $a = 10$, $b = 4$ [$c = 1$, $\gamma = 0$ (a1-b1)], [$c = 4$, $\gamma = 0$ [$c = 1$, $\gamma = 0.002$ (a3-b3)]

..e Numerous studies have reported the chaos suppressing ...ct of external noise in classical systems. Crutchfield *et al* [.] have observed that external noise behaves as a scaling ...able suppressing the chaocity of the system.



...ure 3. Time series and phase plots for Q, P with external noise and η ..., a = 10 b = 4, c = 0 [γ = 0 (a1-b1)], [γ = 0.002 (a2-b2)], [γ = 0.005 ...b3)]



...ure 4. Time series and phase plots for Q, P with external noise and η ... a = 10 b = 4 [c = 1, γ = 0 (a1-b1)], [c = 4, γ = 0 (a2-b2)], [c = 1, γ = 0.002 (a3-b3)]

Our results would seem to endorse this view, however, ...ough a different approach. We have examined the effect of ...ise on a classical system as a limiting case of an open quantum ...stem, whose dynamics have been derived through the ...enfest's theorem and the quantum version of the Langevin ...uation

...e case $\hbar = 1$, which incorporates the quantum effects ...ough a coupling of the P, Q equations with μ , α equations, ...ows the existence of transient chaos in the initial stages of ...e motion, while the long term evolution tends towards ...gularity The length and extent of the initial transient chaotic ...olution increases with the steepness of the potential well ...igure 5)

The existence of bifurcation of motion, as in the classical ...ase, is observed, but at a much later stage of evolution. These ...oints are also diffused, indicating the existence of tunneling, ...d increase with an increase in the potential gradient.

Effects of damping analogous to those observed in the ...lassical case and reported above are also seen and progress towards point attractor observed. The motion of the fluctuation variables μ and α , however, continues to be chaotic as is to be expected since they represent the spreading of the wave packets. When the noise function is superposed on the system, the time series and the phase plots show an increase in the regularity of the motion. Trajectories get confined to one well of the potential. The chaocity of the fluctuation variables also decreases considerably indicating that coherence is lost after a longer interval.

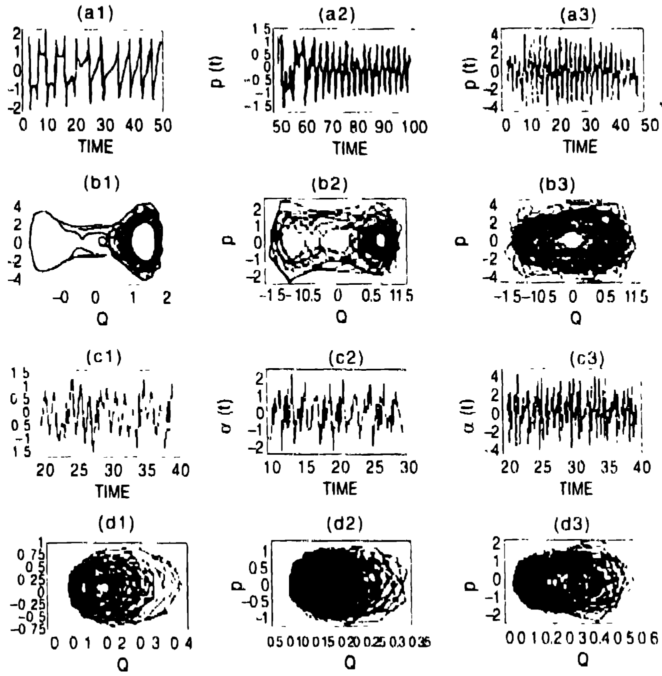


Figure 5. Time series and phase plots for Q, P (a-b) and μ, α (c-d) without external noise and η = 1, a = 10, b = 4 [c = 0, γ = 0.1 (a1 b1, c1-d1)], [c = 1, γ = 0.1 (a2-b2, c2 d2)], [c = 4, γ = 0.1 (a3-b3, c3 d3)]

Conservative quantum systems have been studied [31-33] and observed to generate chaos, which was postulated to arise from the tunnelling effect. The intricate relationship between chaos and quantum tunnelling has been extensively investigated in several studies. In unidimensional systems such effects manifest themselves as classically forbidden penetrations through potential step barriers or as small energy splittings within pairs of symmetric and anti-symmetric eigenstates of the double well potential. In multidimensional systems, the situation is relatively complex in the sense that classically regular and chaotic systems may coexist. The phase space in such situations may consist of chaotic islands filled with distorted tori in an ocean of densely populated invariant tori representing regular motion. One form of tunnelling that may subsist in this phase space is the so called 'chaos assisted tunnelling'. In this mixed phase space, there would exist tori that satisfy the Einstein-Brillouin-Keller EBK quantization condition - such tori would associate with wave functions that satisfy the Schrodinger's time dependent equation. For unidimensional and nearly integrable multidimensional systems such wavefunctions quantizing at or near the same energy would behave as isolated systems However, in systems having a large degree of chaotic behaviour represented by the distortion of the Kol'mogorov-Arnol'd-Moser (KAM) invariant tori, significant influence is exerted by the neighbouring chaotic states on the above wavefunctions and 'crossing over' occurs when a chaotic level passes close to a regular one Additionally, potential functions of many types generate actual wavefunctions that are linear combinations of two or more "pure" wavefunctions (wavefunctions quantizing at or near the same energy) This may result in situations of 'crossing over' from one pure wavefunction to another Tunnelling may also occur between tori inter se (whether a potential barrier subsists or not) in nearly integrable systems that still retain a majority of the invariant tori although distorted and embedded in a sea of feeble resonance and chaotic motion Modelling the tunnelling process as an ensemble of random matrices, it has been shown in [34] that there exists good agreement between predicted and observed splitting distributions, which vindicates the major role played by chaos in the tunnelling process

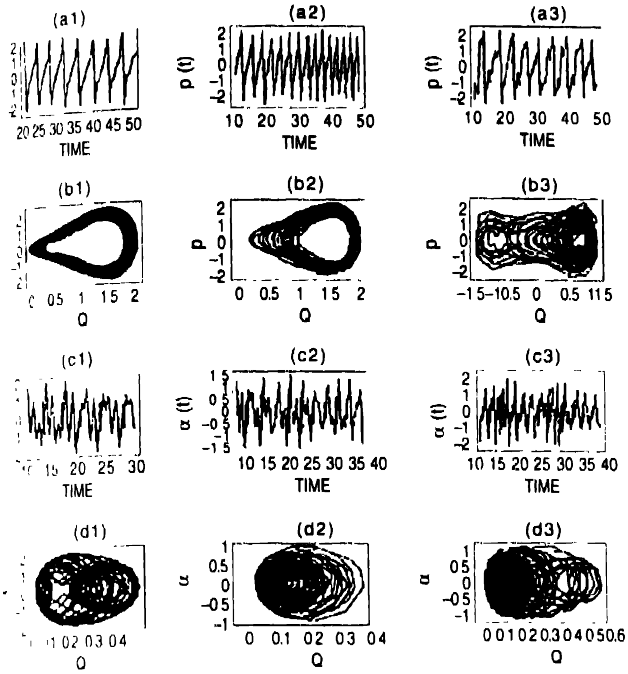
Another related study of significance involves the analysis of the impact of a dynamic external field on quantum systems and its effect on the rate of transitions across potential barriers Using the model of a particle trapped in a double well potential in the presence of a monochromatic external force field and applying an adaptation of the renormalization group technique, analytic expression for the external field amplitude required to facilitate quantum tunnelling has been derived [35]. It is also established that dynamic field induced perturbations result in the creation of a dense set of resonance zones in the phase space of conservative systems These resonance zones tend to remain isolated until the amplitude of the perturbation exceeds a certain threshold level and particles in the vicinity of a resonance zone cannot move through the phase space to the vicinity of another resonance zone However, once the amplitude of the perturbation exceeds a certain value, the KAM invariant tori get distorted and quantum tunnelling is facilitated by the opening of a pathway between neighbouring resonance zones. Destruction of KAM invariants also manifests itself in the phase space acquiring a chaotic look It, therefore, seems clearly evident that the results of [34, 35] are complementary.

Further work on similar lines has been reported in [36]. Here quantum signatures of the KAM transition from regular to chaotic dynamics have been studied Using the methodology of the QTM (Quantum theory of motion), in contrast to the renormalization group technique of the earlier study, both wave and particle pictures of double well oscillator in the presence of an external monochromatic field are investigated Based on the above analysis, it is concluded in this study that quantum fluctuations are enhanced by classical chaos but classical stochasticity is, however, suppressed by the quantum nonclassical effects. A strong coherent oscillatory diffusion between stable KAM tori is reported as a fallout of the simultaneous occurrence of and interaction between quantum tunnelling and classical chaos. Quantum phase portraits exhibit cantorus like islands even with perturbation amplitudes at which classical KAM tori completely breakdown thereby vindicating the suppression of classical chaos by quantum effects The breakdown of the KAM tori is accompanied by opening of a pathway between adjacent resonance zones facilitating the process of quantum tunnelling. A robust coherence is reported to develop due to the interplay of quantum chaos and quantum tunnelling.

Our results seem to lend credence to the above views, even in the case of dissipative systems, since a rise in the steepness of the potential function is causing a significant enhancement of chaocity. In this context, the relationship between dissipation and tunnelling has also been the subject of extensive research [37, 38]. Caldeira and Leggett [37, 38] have, using a variant of the instanton technique, established that quantum tunnelling is invariably suppressed by dissipation and that the suppression factor can be uniquely related to the dissipation coefficient in the case of a linear dissipation mechanism. Our results endorse the above conclusions but through an entirely different mechanism where the dissipation is modelled by means of the system-reservoir theory and further development is through the quantum version of the Langevin equation

The presence of noise is seen to smoothen out chaos in varying degrees in our study. This view corroborates the findings of several earlier studies on the subject, albeit in classical dynamical systems, especially those conducted in the context of the so-called 'stochastic resonance' [39-42] The phenomenon of 'stochastic resonance' essentially personifies the enhancement of the response of a system to a weak input signal by superposing on it a noise signal of appropriate intensity, such response being gauged by measures as spectral power amplification, signal to noise ratio or other input-output correlation measures. The system also exhibits a decrease in the entropy or other measures of disorder In fact, there is considerable evidence that superposition of noise on non linear classical dynamical systems may result in new, more ordered behaviour with more orderly temporal and spatial structures and an increase in the degree of coherence. The impact of weak additive (Langevin) noise on the dynamics of the rf superconducting interference device above its homoclinic threshold have been investigated by Bulsara *et al* using the standard Hamiltonian formalism in [43] and pronounced smoothening effect on the chaotic attractors is reported Elsewhere [44], through an application of time-dependent perturbation techniques to the Fokker Planck equation of the double-well potential, conditions for observance of stochastic resonance are derived and interpreted.

Stochastic resonance has been investigated and observed in the several physical systems like the Schmitt trigger [45], the ring laser [46], passive optical bistable systems [47], Brownian particle [48], tunnel diode [49], ferromagnetic, ferroelectric and ıer magnetic systems [50-53], several chemical systems [54-·], super conducting quantum interference devices [43,57]



re 6 Time series and phase space plots for Q, P (a-b) and μ, α (c-ıth external noise and $\eta = 1$, $a = 10$, $b = 4$ [$c = 0$, $\gamma = 0$ (a1-b1, c1 ;, 0 $\gamma = 0.01$ (a2-b2, c2-d2)], [$c = 1$, $\gamma = 0.01$ (a3-b3, c3-d3)]

Stochastic resonance of a Brownian particle in a double-l potential and indeed, the phenomenon in general, may be lerstood in terms of the existence of different time scales. e first one relates to the inter-well dynamics and the ıations of fluctuations around the stable fixed points; the ond relates to the intra-well dynamics and corresponds to average time of barrier crossings. If the height of the barrier tential is large compared to the noise intensity, a clear aration of the two time scales takes place and the system namics become strongly dependent on the noise level. In the rmal course, under the influence of a small periodic force, nsitions between wells would be prohibited in the absence of ise However, because of the nonlinearity and the nsequential exponential dependence of the rates, a small dulation of the barrier height occurs with the output signal wing a delta peak at its modulation frequency. Periodic nponents are also present in the output at the intra-well scale. restingly, the noise frequency compensates for the mismatch ween the driving frequency and eigenfrequency and a ximal periodic component appears in the output signal for a tain well defined, noise intensity. For such intensity, the rage crossing time across the barrier matches closely with modulation signal period. This causes a time merging ration to take place whereupon even a small modulation of barrier potential causes a remarkable amplification of the put signal

6. Limitations of the study

The use of squeezed coherent states to describe the motion of the particle results in the truncation of the quantum phase space to a semiquantal phase space and may be subject to significant errors after a sufficiently large time scale. In this context, it has been shown [58] that such truncation turns the system's space into an artificially bounded space and thus the non-compactness is lost. Hence, the ability to display the opposite phenomena of chaotic suppression in quantum mechanical systems is lost as well. However, the same assumption has been effectively employed to study quantum chaos without noise [59, 60]

The system reservoir theory employed to model dissipation with the reservoir consisting of harmonic oscillators with a continuous frequency distribution and a linear coupling to the system through a coupling constant that is a smooth function of the oscillator frequencies with energy dissipation being a Markov process is standard in the modeling of dissipative systems, particularly those involving electromagnetic radiations.

Our classical analysis which has been derived in the limit $\hbar \to 0$, is also based on the assumption that for a sufficiently narrow wave packet, the centroid of the quantum state will follow a classical trajectory *i.e* on the validity of the Ehrenfest's theorem to characterize the classical regime In view of the strategic importance of this assumption in our analysis, we examine this aspect in detail.

For the purpose, we start by considering a classical ensemble represented by a probability distribution $\rho(q,p,t)$ in phase space. Then by Liouville's theorem, $\rho(q,p,t)$ satisfies

$$\frac{\partial}{\partial t}\rho(q,p,t) = -\frac{p}{m}\frac{\partial}{\partial q}\rho(q,p,t) - F(q)\frac{\partial}{\partial q}\rho(q,p,t) \qquad (61)$$

(Since the process under investigation involves the limiting case $\hbar \to 0$, a sequence of states of the quantum state function needs to be defined. To facilitate comparison, we make the assumption that the classical limit of the quantum state is an ensemble.)

The averages of the classical position and momentum functions are

$$\langle q \rangle = \iint q\rho(q,p,t)dp\,dp \qquad (62)$$

and

$$\langle p \rangle = \iint p\rho(q,p,t)dq\,dp. \qquad (63)$$

Differentiating these expressions with respect to t, using (61) and integrating by parts, we get,

$$\frac{d}{dt}\langle q \rangle = \frac{\langle q \rangle}{m} \qquad (64)$$

and

$$\frac{d}{dt}\langle q\rangle = \iint F(q)\rho(q,p,t)\,dq\,dp. \tag{65}$$

Expanding (65) as a Taylor's series expansion in powers of $\delta q = q - \langle q\rangle$, we get,

$$\frac{d}{dt}\langle q\rangle = F(\langle q\rangle) + \frac{1}{2}\left\langle(\delta q)^2\right\rangle \frac{\partial^2}{\partial\langle q\rangle^2} F(\langle q\rangle)+\ldots, \tag{66}$$

where

$$\left\langle(\delta q)^2\right\rangle = \iint (\delta q)^2 \rho(q,p,t)\,dq\,dp \tag{67}$$

measures the width of the classical probability distribution. We, therefore, conclude that the centroid of a classical ensemble need not follow a classical trajectory if the width of the probability distribution is not negligible.

Assuming the particle to be moving in a one dimensional scalar potential $V(x)$, its quantum mechanical Hamiltonian operator is $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$ and the equations of motion are

$$\frac{d\hat{q}}{dt} = \frac{\hat{p}}{m} \tag{68}$$

and

$$\frac{d\hat{q}}{dt} = F(\hat{q}). \tag{69}$$

Averaging in some quantum state yields

$$\frac{d}{dt}\langle\hat{q}\rangle = \frac{\langle\hat{p}\rangle}{m} \tag{70}$$

and

$$\frac{d}{dt}\langle\hat{q}\rangle = \langle F(\hat{q})\rangle. \tag{71}$$

Now, if we can approximate the average of the function of position with the function of the average position *i.e.* $\langle F(\hat{q})\rangle = F(\langle\hat{q}\rangle)$, then (71) may be replaced by

$$\frac{d}{dt}\langle\hat{p}\rangle = F(\langle\hat{q}\rangle). \tag{72}$$

As in the classical case, we expand eq. (69) about the centroid as a Taylor's series expansion in powers of $\delta\hat{q} = \hat{q} - \langle\hat{p}\rangle$ and then average in some state, to get

$$\frac{d\langle\hat{p}\rangle}{dt} = F(\langle\hat{q}\rangle) + \frac{1}{2}\left\langle(\delta\hat{q})^2\right\rangle \frac{\partial^2}{\partial\langle\hat{q}\rangle^2} F(\langle\hat{q}\rangle)+\ldots, \tag{73}$$

where $\left\langle(\delta\hat{q})^2\right\rangle$ is a measure of the width of the probabili distribution in configuration space. From eq. (73), it follows th Ehrenfest's theorem holds and $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ obey the classic equations only in the limit that $\left\langle(\delta\hat{q})^2\right\rangle$ and the higher ord terms are negligible *i.e.* when the state function in configurati space is a wave packet with small width.

However, since $\left\langle(\delta\hat{q})^2\right\rangle$ need not invariably vanish in classical limit, we cannot conclude that the terms beyond F(in (73) are quantum mechanical in origin. Furthermore, eqs (6 and (73) are identical in form so that the terms in eq.(73) involv $\delta\hat{q}$ cannot be interpreted as quantum corrections to classi behavior. They merely indicate that the centroid of the quantu probability distribution does not follow a classical trajectory.

Thus, we conclude that the violation of Ehrenfest's theorem expressed by the higher order terms of (73) is not necessarily quantum mechanical origin and a classical ensemble may behav similarly.

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